An Optimized Approach to Linking Attribution Effects Over Time

Attributing the excess return of a portfolio relative to a benchmark is quickly becoming a distinct art form. Although single periods are comparatively easy to analyze, extending the analysis across multiple periods is challenging because distortion effects can lead to misleading results. The author analyzes the problem and offers possible solutions.

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Performance Attribution

In performance attribution, the returns of a portfolio are compared against those of a benchmark, and the relative performance (i.e., excess return) is decomposed into various attribution effects. These attribution effects, in turn, explain the results of active decisions by the portfolio manager. Performance attribution is a rich topic that has been approached from many directions. For example, Fama (1972) proposed a risk-based methodology for comparing portfolio and benchmark returns. Brinson and Fachler (1985) later introduced a performance attribution system that decomposed the excess return into various effects such as issue selection and sector selection. This approach gained widespread use due to its insightful analysis and the straightforward nature of the calculations involved. More recently, Ankrim (1992) considered a risk-adjusted performance attribution scheme in which the basic structure of the Brinson-Fachler method is preserved, but where performance measures are adjusted for risk in the spirit of Fama. Allen (1991) and Ankrim and Hensel (1994) generalized the above methods to account for the effect of currency fluctuations on global portfolios.

In all of the approaches described above, however, the attribution is based on a single-period analysis. Yet, if performance is measured over an extended length of time, say one year, then rebalancing effects may lead to significant inaccuracies within a single-period framework. Therefore, it is imperative to link the single-period attribution effects over multiple periods in a precise and meaningful way.

In arithmetic attribution, the performance of a portfolio relative to a benchmark is given by the difference $R - \bar{R}$, where $R$ and $\bar{R}$ refer to portfolio and benchmark returns, respectively. This relative performance, in turn, is decomposed at the sector level into attribution effects that measure how well the portfolio manager weighted the appropriate sectors and selected securities within the sectors. The sum of the attribution effects gives the relative performance, $R - \bar{R}$.

The basic challenge of arithmetic attribution is to find a robust algorithm to link the attribution effects over time without introducing distortions or unexplained residuals. The purpose of this paper is to present such an approach. Conceptually, our methodology is based on two key points. The first point is to recognize that geometric compounding leads to a characteristic scaling law, which relates the single-period excess returns to the linked excess return. Deducing the correct scaling law ensures that any residual will be small. The second point concerns the optimal distribution of this small residual among the different periods to produce a residual-free linking algorithm. This approach minimizes the distortion that arises from overprinting certain periods relative to others.
SINGLE-PERIOD ARITHMETIC ATTRIBUTION

The portfolio return \( R_t \) for a single period \( t \) can be written as the weighted average return over \( N \) sectors

\[
R_t = \sum_{i=1}^{N} w_{it} r_{it} ,
\]

where \( w_{it} \) and \( r_{it} \) are the portfolio weights and returns for sector \( i \) and period \( t \), respectively. The sectors, in principle, can be based on any market segmentation scheme (e.g., economic sector, industry group, duration, P/E ratio, etc.). Ideally, however, the sectorization scheme should be reflective of the actual investment decision-making process. For the benchmark, the returns are given by

\[
\bar{R}_t = \sum_{i=1}^{N} \bar{w}_{it} \bar{r}_{it} ,
\]

with the overbar denoting the benchmark. The arithmetic measure of relative performance is therefore

\[
R_t - \bar{R}_t = \sum_{i=1}^{N} w_{it} r_{it} - \sum_{i=1}^{N} \bar{w}_{it} \bar{r}_{it} .
\]

By conveniently adding quantities (in brackets, below) equal to zero, this difference can be rewritten

\[
R_t - \bar{R}_t = \sum_{i=1}^{N} w_{it} r_{it} - \sum_{i=1}^{N} \bar{w}_{it} \bar{r}_{it} + \left[ \sum_{i=1}^{N} (w_{it} \bar{r}_{it} - \bar{w}_{it} \bar{r}_{it}) \right] + \left[ \sum_{i=1}^{N} (\bar{w}_{it} - w_{it}) \bar{R}_{it} \right].
\]

Combining terms, we obtain the desired result,

\[
R_t - \bar{R}_t = \sum_{i=1}^{N} w_{it} (r_{it} - \bar{r}_{it}) + \sum_{i=1}^{N} (w_{it} - \bar{w}_{it}) (\bar{r}_{it} - \bar{R}_{it}).
\]

We interpret the terms in the first summation to be the issue selection

\[
I_{it} = w_{it} (r_{it} - \bar{r}_{it}).
\]

Thus, \( I_{it} \) measures how well the portfolio manager picked overperforming securities in sector \( i \) during period \( t \).

Similarly, the terms in the second summation of Equation (5) we interpret to be the sector selection,

\[
S_{it} = (w_{it} - \bar{w}_{it})(\bar{r}_{it} - \bar{R}_{it}),
\]

which measures the extent to which the manager overweighted the overperforming sectors. The total issue selection and sector selection are found by summing the contributions over all sectors. Thus,

\[
I_t = \sum_{i=1}^{N} I_{it}, \quad S_t = \sum_{i=1}^{N} S_{it}.
\]

The above relations allow us to write the relative performance for period \( t \) as

\[
R_t - \bar{R}_t = \sum_{i=1}^{N} (I_{it} + S_{it}) = I_t + S_t.
\]

To summarize, the single-period relative performance has been decomposed into attribution effects at the sector level. The attribution effects combine to give the total excess return for the period, \( R_t - \bar{R}_t \).

MULTIPLE-PERIOD ARITHMETIC ATTRIBUTION

It is desirable to extend the above analysis to the multiple-period case. The portfolio and benchmark returns, geometrically compounded over \( T \) periods, are given by

\[
1 + R = \prod_{i=1}^{T} (1 + R_i), \quad 1 + \bar{R} = \prod_{i=1}^{T} (1 + \bar{R}_i).
\]

Just as we define the relative performance for the single-period case by the difference in single-period returns, it is natural to define it for the multiple-period case as the difference in linked returns, \( R - \bar{R} \). The objective is to link the attribution effects in such a way that they exactly sum to give the relative performance, while minimizing any possible distortions.

If the returns are small, then the relative performance is approximately given by

\[
R - \bar{R} \approx \sum_{i=1}^{T} (R_i - \bar{R}_i).
\]
However, this approximation breaks down for large returns. A better approach is to multiply the right side of Equation (11) (see page 37) by a constant factor $A$ that takes into account the characteristic scaling which arises from geometric compounding:

$$R - \overline{R} \approx A \sum_{i=1}^{T} (R_i - \overline{R}_i).$$

(12)

An obvious possible choice for $A$ is given by

$$A = \frac{R - \overline{R}}{\sum_{j=1}^{T} (R_j - \overline{R}_j)}.$$

(13)

However, this naive solution is unacceptable because it does not necessarily reflect the characteristic scaling of the system. Furthermore, it may easily occur that the numerator and denominator of the above expression have opposite sign. Such a scenario would be counterintuitive because it would imply that positive relative performance from period $t$ contributes negatively to the linked relative performance.

We expect that the scaling that results from geometric compounding can be deduced by relating the actual single-period returns to the mean geometric returns. Therefore, substituting $(1 + R)^{1/T} - 1$ for the single-period returns $R_i$, and similarly for the benchmark, we obtain

$$A = \frac{1}{T} \left[ \frac{(R - \overline{R})}{(1 + R)^{1/T} - (1 + \overline{R})^{1/T}} \right], \quad (R \neq \overline{R}).$$

(14)

Note that $A$ satisfies the required property of being always positive. For the special case $R = \overline{R}$, it is easy to show that the above expression has limiting value

$$A = (1 + R)^{(T-1)/T}, \quad (R = \overline{R}).$$

(15)

Although this choice of $A$ correctly describes the characteristic scaling properties, Equation (12) still leaves a small residual for general sets of returns. However, we can introduce a set of corrective terms $\alpha_t$ that distribute this small residual among the different periods so that the following equation exactly holds

$$R - \overline{R} = \sum_{i=1}^{T} (A + \alpha_t)(R_i - \overline{R}_i).$$

(16)

The problem now reduces to calculating the $\alpha_t$. We believe that the most accurate and intuitive way to link arithmetic attribution effects over time is to weight each period as evenly as possible, so that distortions related to over-weighting certain periods relative to others are avoided. In other words, the $\alpha_t$ should be constructed to be as small as possible, in order that the linking coefficients $A + \alpha_t$ be as uniformly distributed as possible. In order to find the optimal solution, we must minimize the function

$$f = \sum_{i=1}^{T} \alpha_i^2,$$

(17)

subject to the constraint of Equation (16). This is a standard problem involving Lagrange multipliers,$^2$ and the optimal solution is given by

$$\alpha_t = \frac{\left[ R - \overline{R} - A \sum_{j=1}^{T} (R_j - \overline{R}_j) \right]}{\sum_{j=1}^{T} (R_j - \overline{R}_j)^2} (R_i - \overline{R}_i),$$

(18)

where the subscript $j$ is a dummy index of summation. With the $\alpha_t$ thus determined, the linking problem is solved.$^3$ The optimized linking coefficients $\beta_{it}^{opt}$ are thus given by

$$\beta_{it}^{opt} = A + \alpha_t,$$

(19)

with $A$ defined in Equations (14) and (15), and $\alpha_t$ given by Equation (18). Substituting Equation (9) (see page 37) and Equation (19) into Equation (16) we obtain

$$R - \overline{R} = \sum_{i=1}^{T} \sum_{j=1}^{N} \beta_{it}^{opt} (I_{it} + S_{it}).$$

(20)

In summary, we have fully decomposed the linked excess return into attribution effects at the sector level. Summing these quantities over all sectors and time periods gives exactly the linked excess return, $R - \overline{R}$ (i.e., no unexplained residual). Although we have considered only issue selection and sector selection, it is clear that the linking algorithm presented above can be applied just as easily to situations in which there are additional attribution effects (e.g., interaction effect or currency effect).

Another approach to arithmetic linking has recently been discussed in the literature by Carino (1999). In this ap-
A factor $k_t$ for period $t$ is defined by the following relation
\[
\frac{1 + R_t}{1 + \bar{R}_t} = \exp[k_t(R_t - \bar{R}_t)]
\]
(21)

Thus, $k_t$ is given by
\[
k_t = \frac{\ln(1 + R_t) - \ln(1 + \bar{R}_t)}{R_t - \bar{R}_t}.
\]
(22)

The ratio of the linked returns follows immediately from Equation (21)
\[
\frac{1 + R}{1 + \bar{R}} = \prod_{t=1}^{T} \frac{1 + R_t}{1 + \bar{R}_t} = \exp \left[ \sum_{t=1}^{T} k_t(R_t - \bar{R}_t) \right],
\]
(23)

which leads to
\[
\ln(1 + R) - \ln(1 + \bar{R}) = \sum_{t=1}^{T} k_t(R_t - \bar{R}_t).
\]
(24)

Another factor $k$ is defined for the linked returns
\[
k = \frac{\ln(1 + R) - \ln(1 + \bar{R})}{R - \bar{R}},
\]
(25)

which, together with Equation (24), gives
\[
R - \bar{R} = \sum_{t=1}^{T} k_t(R_t - \bar{R}_t).
\]
(26)

Therefore, this logarithmic-based approach leads to linking coefficients $\beta^\log_t$ given by
\[
\beta^\log_t = \left[ \frac{R - \bar{R}}{\ln(1 + R) - \ln(1 + \bar{R})} \right] \frac{\ln(1 + R_t) - \ln(1 + \bar{R}_t)}{R_t - \bar{R}_t}
\]
(27)

It is interesting to compare the two sets of coefficients in greater detail. We performed computational simulations linking single-month attribution effects over a 12-month period. The portfolio and benchmark returns were drawn from normal distributions, with the standard deviation set equal to the absolute value of the mean return. The portfolio and benchmark distributions were kept fixed for the 12-month period, and each data point was calculated by averaging the linking coefficients over 1000 sample paths drawn from the same fixed distributions. The mean monthly returns were then varied from -10% to +20%, in order to obtain a understanding of the global behavior of the linking coefficients. Typical annual returns varied from -70% on the low end to +800% on the high end. In Figure 1 (see page 40) we present the resulting contour plots of the average logarithmic and optimized coefficients. In both cases, the coefficients increase from an average of less than 0.5 for the smallest returns to more than 6.0 for the largest returns. Furthermore, we see that for any combination of portfolio and benchmark returns, the average coefficient is virtually identical in both approaches. Evidently, the reason for this similarity is that the coefficients in the logarithmic algorithm also correctly reflect the scaling properties.

A more interesting study, however, is to compare the standard deviation for both sets of coefficients for the same set of returns used in Figure 1. We first calculate for a single 12-month period $\sigma$, the percent standard deviation of the linking coefficients normalized by the average linking coefficient $\langle \beta \rangle$ for that 12-month period,
\[
\sigma = 100 \frac{\sqrt{\langle \beta^2 \rangle - \langle \beta \rangle^2}}{\langle \beta \rangle}.
\]
(28)

We then average $\sigma$ over 1000 sample paths in order to obtain a good estimate of the average normalized standard deviation of the linking coefficients. The resulting contour plots are presented in Figure 2 (see page 41). Now, we observe fundamentally distinct behavior for the two cases. For the logarithmic coefficients, the normalized standard deviation increases in concentric circles about the origin, rising to over 10% for the largest returns considered here. By contrast, the optimized coefficients exhibit valleys of extremely low standard deviation extending along the directions $R = \pm \bar{R}$. This
property of the standard deviation for the optimized coefficients is very appealing, because typically the portfolio returns are expected to more or less track the benchmark returns. In other words, for the usual case, the optimized coefficients have a much smaller standard deviation than the logarithmic coefficients. From Figure 2 (see page 41), the magnitude of this difference can be easily determined. For instance, we see that for mean monthly portfolio and benchmark returns of 10%, the normalized standard deviation for the logarithmic case is roughly 6%, versus only about 1.5% for the optimized case. Although the results of Figures 1 and 2 (see page 41) were obtained for a 12-month period with specific distributions, we have conducted extensive simulations with different numbers of periods and different distributions and found that the results are completely consistent with those shown in these figures.

It is natural to ask what kinds of differences might arise in practice between the two sets of linking coefficients. In Table 1 (see page 42) we present a hypothetical set of portfolio and benchmark returns for a six-month period, together with the resulting linking coefficients for the logarithmic and optimized cases. We note that the standard deviation of the optimized coefficients is very small, with the coefficients ranging from roughly 1.41 to 1.42. For the logarithmic case, on the other hand, the coefficients vary from 1.26 to 1.54. We observe that the logarithmic coefficients tend to overweight periods with low returns, as discussed above. The linked portfolio and benchmark returns for this example are 64.37% and 39.31%, respectively, for an excess return of 25.06 percent. In Table 1 we also decompose the single-period relative performance into issue selection $I_t$ and sector selection $S_t$. The values were specifically chosen for illustrative purposes with the average single-period issue selection and sector selection being equal. Applying the logarithmic linking algorithm, we find that the linked issue selection is 10.88%, and that the linked sec-
tor selection is 14.18 percent. Using the optimized coefficients, the corresponding values are 12.52% and 12.54%, respectively. In both cases, the issue selection and sector selection add to give the correct relative performance of 25.06%, so that there is no residual in either method. However, the optimized approach more accurately reflects the fact that, on average, the issue selection and sector selection were equal.

CONCLUSIONS

Single-period performance attribution is a useful tool for decomposing excess return into attribution effects that explain the results of active management decisions. Over long periods of time, however, a single-period analysis becomes increasingly inaccurate, and a multiple-period approach is necessary. In this paper, we have presented an intuitive and robust algorithm for linking single-period attribution effects over time. The method consists of first deducing the correct scaling between the single-period excess returns and the linked excess return, and then optimally distributing the residual among the different time periods. This approach minimizes the distortion that arises from overweighting certain periods relative to others.

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REFERENCES


**ENDNOTES**

1 Some analysts prefer to use the benchmark weight in the expression for issue selection, and then include an interaction term. Such a choice has no effect on the linking algorithm discussed in this paper.


3 The only way for the $\alpha_t$ to be singular is if the portfolio return is exactly equal to the benchmark return for each and every period. In such an unusual case, all of the $\alpha_t$ can be set to zero.

4 For some unrealistic combinations of returns, the $\alpha_t$ may oscillate enough to force some of the optimized coefficients to be negative. Nevertheless, our research shows that for any reasonable set of single-period returns (ranging from, say, -70% to +100%), the optimized coefficients are in fact always positive. In any case, the problem of negative linking coefficients is best avoided by simply linking at a higher frequency, which thereby reduces the magnitude of the single-period returns.

5 Although the logarithmic coefficients are intrinsically positive, they exhibit the same large oscillations as the optimized coefficients for extreme sets of single period returns. Again, the best approach for dealing with exceedingly large single-period returns is to simply increase the linking frequency.

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**Table 1**

Comparison of the Logarithmic ($\beta_{t \log}$) and Optimized ($\beta_{t \text{opt}}$) Coefficients for a Hypothetical Six-month Period. Portfolio and Benchmark Returns are given by $R_t$ and $\bar{R}_t$, respectively. Also presented are the Single-period Issue Selection $I_t$ and Sector Selection $S_t$.

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<th>$\bar{R}_t$ (%)</th>
<th>$\beta_{t \log}$</th>
<th>$\beta_{t \text{opt}}$</th>
<th>$I_t$ (%)</th>
<th>$S_t$ (%)</th>
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